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# Strong convergence algorithm for approximating the common solutions of a variational inequality, a mixed equilibrium problem and a hierarchical fixed-point problem

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This paper is dedicated to Yassmine Bnouhachem

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**Abstract**

This paper investigates the common set of solutions of a variational inequality, a mixed equilibrium problem, and a hierarchical fixed-point problem in a Hilbert space. A numerical method is proposed to find the approximate element of this common set. The strong convergence of this method is proved under some conditions. The proposed method is shown to be an improvement and extension of some known results.

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**Keywords:** mixed equilibrium problem; variational inequality problem; hierarchical fixed-point problem; projection method

## 1 Introduction

Let  $H$  be a real Hilbert space, whose inner product and norm are denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ . Let  $C$  be a nonempty closed convex subset of  $H$  and  $A$  be a mapping from  $C$  into  $H$ . A classical variational inequality problem, denoted by  $VI(A, C)$ , is to find a vector  $u \in C$  such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

The solution of  $VI(A, C)$  is denoted by  $\Omega^*$ . It is easy to observe that

$$u^* \in \Omega^* \iff u^* = P_C[u^* - \rho Au^*], \quad \text{where } \rho > 0.$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems; see [1–25]. The fixed-point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually

establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [1] to study the existence of a solution of the variational inequalities.

We introduce the following definitions, which are useful in the following analysis.

**Definition 1.1** The mapping  $T : C \rightarrow H$  is said to be

(a) monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists an  $\alpha > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C;$$

(c)  $\alpha$ -inverse strongly monotone if there exists an  $\alpha > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e)  $k$ -Lipschitz continuous if there exists a constant  $k > 0$  such that

$$\|Tx - Ty\| \leq k \|x - y\|, \quad \forall x, y \in C;$$

(f) contraction on  $C$  if there exists a constant  $0 \leq k < 1$  such that

$$\|Tx - Ty\| \leq k \|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every  $\alpha$ -inverse strongly monotone  $T$  is monotone and Lipschitz continuous. A mapping  $T : C \rightarrow H$  is called  $k$ -strict pseudo-contraction if there exists a constant  $0 \leq k < 1$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \quad (1.2)$$

The fixed-point problem for the mapping  $T$  is to find  $x \in C$  such that

$$Tx = x. \quad (1.3)$$

We denote by  $F(T)$  the set of solutions of (1.3). It is well known that the class of strict pseudo-contractions includes the class of Lipschitzian mappings, then  $F(T)$  is closed and convex and  $P_{F(T)}$  is well defined (see [2]).

The mixed equilibrium problem, denoted by  $MEP$ , is to find  $x \in C$  such that

$$F_1(x, y) + \langle Dx, y - x \rangle \geq 0, \quad \forall y \in C, \quad (1.4)$$

where  $F_1 : C \times C \rightarrow \mathbb{R}$  is a bifunction, and  $D : C \rightarrow H$  is a nonlinear mapping. This problem was introduced and studied by Moudafi and Théra [3] and Moudafi [4]. The set of solutions of (1.4) is denoted by

$$MEP(F_1) := \{x \in C : F_1(x, y) + \langle Dx, y - x \rangle \geq 0, \forall y \in C\}. \quad (1.5)$$

If  $D = 0$ , then it is reduced to the equilibrium problem, which is to find  $x \in C$  such that

$$F_1(x, y) \geq 0, \quad \forall y \in C. \quad (1.6)$$

The solution set of (1.6) is denoted by  $EP(F_1)$ . Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.6); see [5–9]. In 1997, Combettes and Hirstoaga [10] introduced an iterative scheme of finding the best approximation to the initial data when  $EP(F)$  is nonempty. Recently Plubtieng and Punpaeng [7] introduced an iterative method for finding the common element of the set  $F(T) \cap \Omega^* \cap EP(F_1)$ .

Let  $S : C \rightarrow H$  be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find  $x \in F(T)$  such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \quad (1.7)$$

It is known that the hierarchical fixed-point problem (1.7) links with some monotone variational inequalities and convex programming problems; see [11, 12, 26]. Various methods have been proposed to solve the hierarchical fixed-point problem; see Moudafi [13], Mainge and Moudafi in [14], Marino and Xu in [15] and Cianciaruso *et al.* [16]. Very recently, Yao *et al.* [12] introduced the following strong convergence iterative algorithm to solve problem (1.7):

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 0, \end{aligned} \quad (1.8)$$

where  $f : C \rightarrow H$  is a contraction mapping, and  $\{\alpha_n\}$  and  $\{\beta_n\}$  are two sequences in  $(0, 1)$ . Under some certain restrictions on the parameters, Yao *et al.* proved that the sequence  $\{x_n\}$  generated by (1.8) converges strongly to  $z \in F(T)$ , which is the unique solution of the following variational inequality:

$$\langle (I - f)z, y - z \rangle \geq 0, \quad \forall y \in F(T). \quad (1.9)$$

In 2011, Ceng *et al.* [17] investigated the following iterative method:

$$x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \quad (1.10)$$

where  $U$  is a Lipschitzian mapping, and  $F$  is a Lipschitzian and strongly monotone mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence  $\{x_n\}$  generated by (1.10) converges strongly to the unique solution of the variational inequality

$$\langle \rho U(z) - \mu F(z), x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(T).$$

In this paper, motivated by the work of Yao *et al.* [12], Ceng *et al.* [17], Bnouhachem [18, 19] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.4), and (1.7) in a real Hilbert space. We establish a strong convergence theorem based on this method. We would like to mention that our proposed method is quite general and flexible and includes many known results for solving equilibrium problems, variational inequality problems, and hierarchical fixed-point problems; see, *e.g.*, [11, 12, 14–17, 24, 25] and relevant references cited therein.

## 2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto  $C$ .

**Lemma 2.1** *Let  $P_C$  denote the projection of  $H$  onto  $C$ . Then we have the following inequalities:*

$$\langle z - P_C[z], P_C[z] - v \rangle \geq 0, \quad \forall z \in H, v \in C; \quad (2.1)$$

$$\langle u - v, P_C[u] - P_C[v] \rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \quad (2.2)$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H; \quad (2.3)$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \quad (2.4)$$

**Lemma 2.2** [20] *Let  $F_1 : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying the following assumptions:*

- (i)  $F_1(x, x) = 0, \forall x \in C$ ;
- (ii)  $F_1$  is monotone, *i.e.*,  $F_1(x, y) + F_1(y, x) \leq 0, \forall x, y \in C$ ;
- (iii) for each  $x, y, z \in C$ ,  $\lim_{t \rightarrow 0} F_1(tz + (1-t)x, y) \leq F_1(x, y)$ ;
- (iv) for each  $x \in C, y \rightarrow F_1(x, y)$  is convex and lower semicontinuous.

*Let  $r > 0$  and  $x \in H$ . Then there exists  $z \in C$  such that*

$$F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

**Lemma 2.3** [10] *Assume that  $F_1 : C \times C \rightarrow \mathbb{R}$  satisfies assumptions (i)-(iv) of Lemma 2.2, and for  $r > 0$  and  $\forall x \in H$ , define a mapping  $T_r : H \rightarrow C$  as follows:*

$$T_r(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

*Then the following hold:*

- (i)  $T_r$  is single-valued;
- (ii)  $T_r$  is firmly nonexpansive, *i.e.*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H;$$

- (iii)  $F(T_r) = EP(F_1)$ ;
- (iv)  $EP(F_1)$  is closed and convex.

**Lemma 2.4** [21] *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . If  $T : C \rightarrow C$  is a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ , then the mapping  $I - T$  is demiclosed at 0, i.e., if  $\{x_n\}$  is a sequence in  $C$  weakly converging to  $x$  and if  $\{(I - T)x_n\}$  converges strongly to 0, then  $(I - T)x = 0$ .*

**Lemma 2.5** [17] *Let  $U : C \rightarrow H$  be a  $\tau$ -Lipschitzian mapping, and let  $F : C \rightarrow H$  be a  $k$ -Lipschitzian and  $\eta$ -strongly monotone mapping, then for  $0 \leq \rho\tau < \mu\eta$ ,  $\mu F - \rho U$  is  $\mu\eta - \rho\tau$ -strongly monotone, i.e.,*

$$\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y \rangle \geq (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x, y \in C.$$

**Lemma 2.6** [22] *Suppose that  $\lambda \in (0, 1)$  and  $\mu > 0$ . Let  $F : C \rightarrow H$  be a  $k$ -Lipschitzian and  $\eta$ -strongly monotone operator. In association with a nonexpansive mapping  $T : C \rightarrow C$ , define the mapping  $T^\lambda : C \rightarrow H$  by*

$$T^\lambda x = Tx - \lambda\mu FT(x), \quad \forall x \in C.$$

*Then  $T^\lambda$  is a contraction provided  $\mu < \frac{2\eta}{k^2}$ , that is,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

*where  $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ .*

**Lemma 2.7** [23] *Assume that  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

*where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$ , and  $\delta_n$  is a sequence such that*

- (1)  $\sum_{n=1}^{\infty} \gamma_n = \infty$ ;
- (2)  $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

*Then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

**Lemma 2.8** [27] *Let  $C$  be a closed convex subset of  $H$ . Let  $\{x_n\}$  be a bounded sequence in  $H$ . Assume that*

- (i) *the weak  $w$ -limit set  $w_w(x_n) \subset C$ , where  $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ ;*
- (ii) *for each  $z \in C$ ,  $\lim_{n \rightarrow \infty} \|x_n - z\|$  exists.*

*Then  $\{x_n\}$  is weakly convergent to a point in  $C$ .*

### 3 The proposed method and some properties

In this section, we suggest and analyze our method for finding the common solutions of the variational inequality (1.1), the mixed equilibrium problem (1.4), and the hierarchical fixed-point problem (1.7).

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $D, A : C \rightarrow H$  be  $\theta, \alpha$ -inverse strongly monotone mappings, respectively. Let  $F_1 : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying assumptions (i)-(iv) of Lemma 2.2 and  $S, T : C \rightarrow C$  be nonexpansive mappings such that  $F(T) \cap \Omega^* \cap \text{MEP}(F_1) \neq \emptyset$ . Let  $F : C \rightarrow C$  be a  $k$ -Lipschitzian mapping and be  $\eta$ -strongly monotone, and let  $U : C \rightarrow C$  be a  $\tau$ -Lipschitzian mapping.

**Algorithm 3.1** For an arbitrary given  $x_0 \in C$ , let the iterative sequences  $\{u_n\}$ ,  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated by

$$\begin{aligned} F_1(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C; \\ z_n &= P_C[u_n - \lambda_n A u_n]; \\ y_n &= \beta_n S x_n + (1 - \beta_n) z_n; \\ x_{n+1} &= P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \end{aligned} \quad (3.1)$$

where  $\{\lambda_n\} \subset (0, 2\alpha)$ ,  $\{r_n\} \subset (0, 2\theta)$ . Suppose that the parameters satisfy  $0 < \mu < \frac{2\eta}{k^2}$ ,  $0 \leq \rho\tau < v$ , where  $v = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ . Also,  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$ ,
- (c)  $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ ,
- (d)  $\liminf_{n \rightarrow \infty} r_n > 0$  and  $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$ ,
- (e)  $\liminf_{n \rightarrow \infty} \lambda_n < \limsup_{n \rightarrow \infty} \lambda_n < 2\alpha$  and  $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$ .

**Remark 3.1** Our method can be viewed as an extension and improvement for some well-known results, for example, the following.

- If  $A = 0$ , we obtain an extension and improvement of the method of Wang and Xu [24] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- If we have the Lipschitzian mapping  $U = f$ ,  $F = I$ ,  $\rho = \mu = 1$ , and  $A = 0$ , we obtain an extension and improvement of the method of Yao *et al.* [12] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- The contractive mapping  $f$  with a coefficient  $\alpha \in [0, 1)$  in other papers [12, 15, 22, 25] is extended to the cases of the Lipschitzian mapping  $U$  with a coefficient constant  $\gamma \in [0, \infty)$ .

This shows that Algorithm 3.1 is quite general and unifying.

**Lemma 3.1** Let  $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$ . Then  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{z_n\}$ , and  $\{y_n\}$  are bounded.

*Proof* First, we show that the mapping  $(I - r_n D)$  is nonexpansive. For any  $x, y \in C$ ,

$$\begin{aligned} \|(I - r_n D)x - (I - r_n D)y\|^2 &= \|(x - y) - r_n(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2r_n \langle x - y, Dx - Dy \rangle + r_n^2 \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - r_n(2\theta - r_n) \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Similarly, we can show that the mapping  $(I - \lambda_n A)$  is nonexpansive. It follows from Lemma 2.3 that  $u_n = T_{r_n}(x_n - r_n D x_n)$ . Let  $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$ ; we have  $x^* =$

$$T_{r_n}(x^* - r_n D x^*).$$

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(x^* - r_n D x^*)\|^2 \\ &\leq \|(x_n - r_n D x_n) - (x^* - r_n D x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|D x_n - D x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \quad (3.2)$$

Since the mapping  $A$  is  $\alpha$ -inverse strongly monotone, we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n A u_n] - P_C[x^* - \lambda_n A x^*]\|^2 \\ &\leq \|u_n - x^* - \lambda_n(A u_n - A x^*)\|^2 \\ &\leq \|u_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|A u_n - A x^*\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \quad (3.3)$$

We define  $V_n = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))$ . Next, we prove that the sequence  $\{x_n\}$  is bounded, and without loss of generality we can assume that  $\beta_n \leq \alpha_n$  for all  $n \geq 1$ . From (3.1), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|P_C[V_n] - P_C[x^*]\| \\ &\leq \|\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n)) - x^*\| \\ &\leq \alpha_n \|\rho U(x_n) - \mu F(x^*)\| + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\ &= \alpha_n \|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)x^*\| \\ &\quad + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\ &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + (1 - \alpha_n v) \|y_n - x^*\| \\ &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| \\ &\quad + (1 - \alpha_n v) \|\beta_n S x_n + (1 - \beta_n) z_n - x^*\| \\ &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| \\ &\quad + (1 - \alpha_n v) (\beta_n \|S x_n - S x^*\| + \beta_n \|S x^* - x^*\| + (1 - \beta_n) \|z_n - x^*\|) \\ &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| \\ &\quad + (1 - \alpha_n v) (\beta_n \|S x_n - S x^*\| + \beta_n \|S x^* - x^*\| + (1 - \beta_n) \|x_n - x^*\|) \\ &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| \\ &\quad + (1 - \alpha_n v) (\beta_n \|x_n - x^*\| + \beta_n \|S x^* - x^*\| + (1 - \beta_n) \|x_n - x^*\|) \\ &= (1 - \alpha_n(v - \rho \tau)) \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + (1 - \alpha_n v) \beta_n \|S x^* - x^*\| \\ &\leq (1 - \alpha_n(v - \rho \tau)) \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \beta_n \|S x^* - x^*\| \\ &\leq (1 - \alpha_n(v - \rho \tau)) \|x_n - x^*\| + \alpha_n (\|(\rho U - \mu F)x^*\| + \|S x^* - x^*\|) \end{aligned}$$

$$\begin{aligned}
 &= (1 - \alpha_n(v - \rho\tau)) \|x_n - x^*\| + \frac{\alpha_n(v - \rho\tau)}{v - \rho\tau} (\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \\
 &\leq \max \left\{ \|x_n - x^*\|, \frac{1}{v - \rho\tau} (\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \right\},
 \end{aligned}$$

where the third inequality follows from Lemma 2.6.

By induction on  $n$ , we obtain  $\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \frac{1}{v - \rho\tau} (\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\}$  for  $n \geq 0$  and  $x_0 \in C$ . Hence,  $\{x_n\}$  is bounded and, consequently, we deduce that  $\{u_n\}$ ,  $\{z_n\}$ ,  $\{v_n\}$ ,  $\{y_n\}$ ,  $\{S(x_n)\}$ ,  $\{T(x_n)\}$ ,  $\{F(T(y_n))\}$ , and  $\{U(x_n)\}$  are bounded.  $\square$

**Lemma 3.2** *Let  $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$  and  $\{x_n\}$  be the sequence generated by Algorithm 3.1. Then we have:*

- (a)  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .
- (b) The weak  $w$ -limit set  $w_w(x_n) \subset F(T)$ ,  $(w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\})$ .

*Proof* From the nonexpansivity of the mapping  $(I - \lambda_n A)$  and  $P_C$ , we have

$$\begin{aligned}
 \|z_n - z_{n-1}\| &\leq \|(u_n - \lambda_n A u_n) - (u_{n-1} - \lambda_{n-1} A u_{n-1})\| \\
 &= \|(u_n - u_{n-1}) - \lambda_n (A u_n - A u_{n-1}) - (\lambda_n - \lambda_{n-1}) A u_{n-1}\| \\
 &\leq \|(u_n - u_{n-1}) - \lambda_n (A u_n - A u_{n-1})\| + |\lambda_n - \lambda_{n-1}| \|A u_{n-1}\| \\
 &\leq \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|A u_{n-1}\|.
 \end{aligned} \tag{3.4}$$

Next, we estimate that

$$\begin{aligned}
 \|y_n - y_{n-1}\| &= \|\beta_n S x_n + (1 - \beta_n) z_n - (\beta_{n-1} S x_{n-1} + (1 - \beta_{n-1}) z_{n-1})\| \\
 &= \|\beta_n (S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1}) S x_{n-1} \\
 &\quad + (1 - \beta_n) (z_n - z_{n-1}) + (\beta_{n-1} - \beta_n) z_{n-1}\| \\
 &\leq \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \|z_n - z_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|z_{n-1}\|).
 \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
 \|y_n - y_{n-1}\| &\leq \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \{ \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|A u_{n-1}\| \} \\
 &\quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|z_{n-1}\|).
 \end{aligned} \tag{3.6}$$

On the other hand,  $u_n = T_{r_n}(x_n - r_n D x_n)$  and  $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1} D x_{n-1})$ , we have

$$F_1(u_n, y) + \langle D x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C \tag{3.7}$$

and

$$F_1(u_{n-1}, y) + \langle D x_{n-1}, y - u_{n-1} \rangle + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$



Take  $y = u_{n-1}$  in (3.7) and  $y = u_n$  in (3.8), we get

$$F_1(u_n, u_{n-1}) + \langle Dx_n, u_{n-1} - u_n \rangle + \frac{1}{r_n} \langle u_{n-1} - u_n, u_n - x_n \rangle \geq 0 \quad (3.9)$$

and

$$F_1(u_{n-1}, u_n) + \langle Dx_{n-1}, u_n - u_{n-1} \rangle + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0. \quad (3.10)$$

Adding (3.9) and (3.10) and using the monotonicity of  $F_1$ , we have

$$\langle Dx_{n-1} - Dx_n, u_n - u_{n-1} \rangle + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0,$$

which implies that

$$\begin{aligned} 0 &\leq \left\langle u_n - u_{n-1}, r_n(Dx_{n-1} - Dx_n) + \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n) \right\rangle \\ &= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} \right. \\ &\quad \left. + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1} \right\rangle \\ &= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1} \right\rangle \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) + (x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n) \right\rangle \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &\leq \|u_{n-1} - u_n\| \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| + \|(x_{n-1} - r_n Dx_{n-1}) - (x_n - r_n Dx_n)\| \right\} \\ &\quad - \|u_n - u_{n-1}\|^2 \\ &\leq \|u_{n-1} - u_n\| \left\{ \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \right\} - \|u_n - u_{n-1}\|^2, \end{aligned}$$

and then

$$\|u_{n-1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, let us assume that there exists a real number  $\mu$  such that  $r_n > \mu > 0$  for all positive integers  $n$ . Then we get

$$\|u_{n-1} - u_n\| \leq \|x_{n-1} - x_n\| + \frac{1}{\mu} |r_{n-1} - r_n| \|u_{n-1} - x_{n-1}\|. \quad (3.11)$$

It follows from (3.6) and (3.11) that

$$\|y_n - y_{n-1}\| \leq \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \left\{ \|x_n - x_{n-1}\| + \frac{1}{\mu} |r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| \right\}$$

$$\begin{aligned}
 & + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \Big\} + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|z_{n-1}\|) \\
 = & \|x_n - x_{n-1}\| + (1 - \beta_n) \left\{ \frac{1}{\mu} |r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \right\} \\
 & + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|z_{n-1}\|). \tag{3.12}
 \end{aligned}$$

Next, we estimate that

$$\begin{aligned}
 \|x_{n+1} - x_n\| & = \|P_C[V_n] - P_C[V_{n-1}]\| \\
 & \leq \|\alpha_n \rho (U(x_n) - U(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \rho U(x_{n-1}) + (I - \alpha_n \mu F)(T(y_n)) \\
 & \quad - (I - \alpha_n \mu F)T(y_{n-1}) + (I - \alpha_n \mu F)(T(y_{n-1})) - (I - \alpha_{n-1} \mu F)(T(y_{n-1}))\| \\
 & \leq \alpha_n \rho \tau \|x_n - x_{n-1}\| + (1 - \alpha_n v) \|y_n - y_{n-1}\| \\
 & \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|), \tag{3.13}
 \end{aligned}$$

where the second inequality follows from Lemma 2.6. From (3.12) and (3.13), we have

$$\begin{aligned}
 \|x_{n+1} - x_n\| & \leq \alpha_n \rho \tau \|x_n - x_{n-1}\| + (1 - \alpha_n v) \\
 & \quad \times \left( \|x_n - x_{n-1}\| + \frac{1}{\mu} |r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \right) \\
 & \quad + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|z_{n-1}\|) \\
 & \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \\
 & \leq (1 - (v - \rho \tau) \alpha_n) \|x_n - x_{n-1}\| \\
 & \quad + \frac{1}{\mu} |r_n - r_{n-1}| \|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Au_{n-1}\| \\
 & \quad + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|z_{n-1}\|) \\
 & \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \\
 & \leq (1 - (v - \rho \tau) \alpha_n) \|x_n - x_{n-1}\| \\
 & \quad + M \left( \frac{1}{\mu} |r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| \right). \tag{3.14}
 \end{aligned}$$

Here

$$\begin{aligned}
 M = \max \Big\{ & \sup_{n \geq 1} \|u_{n-1} - x_{n-1}\|, \sup_{n \geq 1} \|Au_{n-1}\|, \sup_{n \geq 1} (\|Sx_{n-1}\| + \|z_{n-1}\|), \\
 & \sup_{n \geq 1} (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \Big\}.
 \end{aligned}$$

It follows by conditions (a)-(e) of Algorithm 3.1 and Lemma 2.7 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Next, we show that  $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$ . Since  $x^* \in F(T) \cap \Omega^* \cap MEP(F_1)$ , by using (3.2) and (3.3), we obtain

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &= \langle P_C(V_n) - x^*, x_{n+1} - x^* \rangle \\
 &= \langle P_C(V_n) - V_n, P_C(V_n) - x^* \rangle + \langle V_n - x^*, x_{n+1} - x^* \rangle \\
 &\leq \langle \alpha_n(\rho U(x_n) - \mu F(x^*)) + (I - \alpha_n \mu F)(T(y_n)) \\
 &\quad - (I - \alpha_n \mu F)(T(x^*)), x_{n+1} - x^* \rangle \\
 &= \langle \alpha_n \rho(U(x_n) - U(x^*)), x_{n+1} - x^* \rangle + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \langle (I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)(T(x^*)), x_{n+1} - x^* \rangle \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + (1 - \alpha_n \nu) \|y_n - x^*\| \|x_{n+1} - x^*\| \\
 &\leq \frac{\alpha_n \rho \tau}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \nu)}{2} (\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
 &\leq \frac{(1 - \alpha_n(\nu - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \nu)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \|z_n - x^*\|^2) \\
 &\leq \frac{(1 - \alpha_n(\nu - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle + \frac{(1 - \alpha_n \nu) \beta_n}{2} \|Sx_n - x^*\|^2 \\
 &\quad + \frac{(1 - \alpha_n \nu)(1 - \beta_n)}{2} \{ \|x_n - x^*\|^2 - r_n(2\theta - r_n) \|Dx_n - Dx^*\|^2 \\
 &\quad - \lambda_n(2\alpha - \lambda_n) \|Au_n - Ax^*\|^2 \}, \tag{3.15}
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\nu - \rho \tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \nu) \beta_n}{1 + \alpha_n(\nu - \rho \tau)} \|Sx_n - x^*\|^2 \\
 &\quad + \frac{(1 - \alpha_n \nu)(1 - \beta_n)}{1 + \alpha_n(\nu - \rho \tau)} \{ \|x_n - x^*\|^2 - r_n(2\theta - r_n) \|Dx_n - Dx^*\|^2 \\
 &\quad - \lambda_n(2\alpha - \lambda_n) \|Au_n - Ax^*\|^2 \} \\
 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\nu - \rho \tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle
 \end{aligned}$$

$$\begin{aligned}
 & + \|x_n - x^*\|^2 + \frac{(1 - \alpha_n v) \beta_n}{1 + \alpha_n (v - \rho \tau)} \|Sx_n - x^*\|^2 \\
 & - \frac{(1 - \alpha_n v)(1 - \beta_n)}{1 + \alpha_n (v - \rho \tau)} \{r_n(2\theta - r_n) \|Dx_n - Dx^*\|^2 \\
 & + \lambda_n(2\alpha - \lambda_n) \|Au_n - Ax^*\|^2\}.
 \end{aligned}$$

Then, from the inequality above, we get

$$\begin{aligned}
 & \frac{(1 - \alpha_n v)(1 - \beta_n)}{1 + \alpha_n (v - \rho \tau)} \{r_n(2\theta - r_n) \|Dx_n - Dx^*\|^2 + \lambda_n(2\alpha - \lambda_n) \|Au_n - Ax^*\|^2\} \\
 & \leq \frac{\alpha_n \rho \tau}{1 + \alpha_n (v - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n (v - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 & \quad + \beta_n \|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
 & \leq \frac{\alpha_n \rho \tau}{1 + \alpha_n (v - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n (v - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 & \quad + \beta_n \|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
 \end{aligned}$$

Since  $\liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2\alpha$ ,  $\{r_n\} \subset (0, 2\theta)$ ,  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\alpha_n \rightarrow 0$ , and  $\beta_n \rightarrow 0$ , we obtain  $\lim_{n \rightarrow \infty} \|Dx_n - Dx^*\| = 0$  and  $\lim_{n \rightarrow \infty} \|Au_n - Ax^*\| = 0$ .

Since  $T_{r_n}$  is firmly nonexpansive, we have

$$\begin{aligned}
 \|u_n - x^*\|^2 & = \|T_{r_n}(x_n - r_n Dx_n) - T_{r_n}(x^* - r_n Dx^*)\|^2 \\
 & \leq \langle u_n - x^*, (x_n - r_n Dx_n) - (x^* - r_n Dx^*) \rangle \\
 & = \frac{1}{2} \{ \|u_n - x^*\|^2 + \|(x_n - r_n Dx_n) - (x^* - r_n Dx^*)\|^2 \\
 & \quad - \|u_n - x^* - [(x_n - r_n Dx_n) - (x^* - r_n Dx^*)]\|^2 \}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \|u_n - x^*\|^2 & \leq \|(x_n - r_n Dx_n) - (x^* - r_n Dx^*)\|^2 - \|u_n - x_n + r_n(Dx_n - Dx^*)\|^2 \\
 & \leq \|x_n - x^*\|^2 - \|u_n - x_n + r_n(Dx_n - Dx^*)\|^2 \\
 & \leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2r_n \|u_n - x_n\| \|Dx_n - Dx^*\|.
 \end{aligned}$$

From (3.15), (3.3), and the inequality above, we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 & \leq \frac{(1 - \alpha_n(v - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 & \quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 & \quad + \frac{(1 - \alpha_n v)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \|z_n - x^*\|^2) \\
 & \leq \frac{(1 - \alpha_n(v - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 & \quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 & \quad + \frac{(1 - \alpha_n v)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \|u_n - x^*\|^2)
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{(1-\alpha_n(\nu-\rho\tau))}{2} \|x_{n+1}-x^*\|^2 + \frac{\alpha_n\rho\tau}{2} \|x_n-x^*\|^2 \\ &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^* \rangle \\ &\quad + \frac{(1-\alpha_n\nu)}{2} \{ \beta_n \|Sx_n-x^*\|^2 + (1-\beta_n)(\|x_n-x^*\|^2 - \|u_n-x_n\|^2 \\ &\quad + 2r_n\|u_n-x_n\| \|Dx_n-Dx^*\|) \}, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)} \|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^* \rangle \\ &\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)} \|Sx_n-x^*\|^2 \\ &\quad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)} \{ \|x_n-x^*\|^2 - \|u_n-x_n\|^2 \\ &\quad + 2r_n\|u_n-x_n\| \|Dx_n-Dx^*\| \} \\ &\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)} \|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^* \rangle \\ &\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)} \|Sx_n-x^*\|^2 \\ &\quad + \|x_n-x^*\|^2 + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)} \{ -\|u_n-x_n\|^2 \\ &\quad + 2r_n\|u_n-x_n\| \|Dx_n-Dx^*\| \}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)} \|u_n-x_n\|^2 &\leq \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)} \|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^* \rangle \\ &\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)} \|Sx_n-x^*\|^2 \\ &\quad + \frac{2(1-\alpha_n\nu)(1-\beta_n)r_n}{1+\alpha_n(\nu-\rho\tau)} \|u_n-x_n\| \|Dx_n-Dx^*\| \\ &\quad + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &= \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)} \|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1}-x^* \rangle \\ &\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)} \|Sx_n-x^*\|^2 \end{aligned}$$

$$+ \frac{2(1-\alpha_n v)(1-\beta_n)r_n}{1+\alpha_n(v-\rho\tau)} \|u_n - x_n\| \|Dx_n - Dx^*\| \\ + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.$$

Since  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$ , and  $\lim_{n \rightarrow \infty} \|Dx_n - Dx^*\| = 0$ , we obtain

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.16)$$

From (2.2), we get

$$\begin{aligned} \|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n Au_n] - P_C[x^* - \lambda_n Ax^*]\|^2 \\ &\leq \langle z_n - x^*, (u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*) \rangle \\ &= \frac{1}{2} \{ \|z_n - x^*\|^2 + \|u_n - x^* - \lambda_n (Au_n - Ax^*)\|^2 \\ &\quad - \|u_n - x^* - \lambda_n (Au_n - Ax^*) - (z_n - x^*)\|^2 \} \\ &\leq \frac{1}{2} \{ \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n - \lambda_n (Au_n - Ax^*)\|^2 \} \\ &\leq \frac{1}{2} \{ \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle u_n - z_n, Au_n - Ax^* \rangle \} \\ &\leq \frac{1}{2} \{ \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\| \}. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\| \\ &\leq \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\|. \end{aligned}$$

From (3.15) and the inequality above, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{(1-\alpha_n(v-\rho\tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\ &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\ &\quad + \frac{(1-\alpha_n v)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1-\beta_n) \|z_n - x^*\|^2) \\ &\leq \frac{(1-\alpha_n(v-\rho\tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\ &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\ &\quad + \frac{(1-\alpha_n v)}{2} \{ \beta_n \|Sx_n - x^*\|^2 + (1-\beta_n) (\|x_n - x^*\|^2 \\ &\quad - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\|) \}, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{\alpha_n \rho \tau}{1+\alpha_n(v-\rho\tau)} \|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(v-\rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \end{aligned}$$

$$\begin{aligned}
& + \frac{(1 - \alpha_n v) \beta_n}{1 + \alpha_n (v - \rho \tau)} \|Sx_n - x^*\|^2 \\
& + \frac{(1 - \alpha_n v)(1 - \beta_n)}{1 + \alpha_n (v - \rho \tau)} \{ \|x_n - x^*\|^2 - \|u_n - z_n\|^2 \\
& + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\| \}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{(1 - \alpha_n v)(1 - \beta_n)}{1 + \alpha_n (v - \rho \tau)} \|u_n - z_n\|^2 \\
& \leq \frac{\alpha_n \rho \tau}{1 + \alpha_n (v - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n (v - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
& \quad + \frac{(1 - \alpha_n v) \beta_n}{1 + \alpha_n (v - \rho \tau)} \|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& \quad + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\| \\
& = \frac{\alpha_n \rho \tau}{1 + \alpha_n (v - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n (v - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
& \quad + \frac{(1 - \alpha_n v) \beta_n}{1 + \alpha_n (v - \rho \tau)} \|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\
& \quad + 2\lambda_n \|u_n - z_n\| \|Au_n - Ax^*\|.
\end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$ , and  $\lim_{n \rightarrow \infty} \|Au_n - Ax^*\| = 0$ , we obtain

$$\lim_{n \rightarrow \infty} \|u_n - z_n\| = 0. \quad (3.17)$$

It follows from (3.16) and (3.17) that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (3.18)$$

Since  $T(x_n) \in C$ , we have

$$\begin{aligned}
\|x_n - T(x_n)\| & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\
& = \|x_n - x_{n+1}\| + \|P_C[V_n] - P_C[T(x_n)]\| \\
& \leq \|x_n - x_{n+1}\| + \|\alpha_n (\rho U(x_n) - \mu F(T(y_n))) + T(y_n) - T(x_n)\| \\
& \leq \|x_n - x_{n+1}\| + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| + \|y_n - x_n\| \\
& \leq \|x_n - x_{n+1}\| + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| + \|\beta_n Sx_n + (1 - \beta_n)z_n - x_n\| \\
& \leq \|x_n - x_{n+1}\| + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| \\
& \quad + \beta_n \|Sx_n - x_n\| + (1 - \beta_n) \|z_n - x_n\|.
\end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$ , and  $\|\rho U(x_n) - \mu F(T(y_n))\|$  and  $\|Sx_n - x_n\|$  are bounded, and  $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ , we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T(x_n)\| = 0.$$

Since  $\{x_n\}$  is bounded, without loss of generality we can assume that  $x_n \rightharpoonup x^* \in C$ . It follows from Lemma 2.4 that  $x^* \in F(T)$ . Therefore,  $w_w(x_n) \subset F(T)$ .  $\square$

**Theorem 3.1** *The sequence  $\{x_n\}$  generated by Algorithm 3.1 converges strongly to  $z$ , which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in \Omega^* \cap MEP(F_1) \cap F(T). \quad (3.19)$$

*Proof* Since  $\{x_n\}$  is bounded  $x_n \rightharpoonup w$  and from Lemma 3.2, we have  $w \in F(T)$ . Next, we show that  $w \in MEP(F_1)$ . Since  $u_n = T_{r_n}(x_n - r_n D x_n)$ , we have

$$F_1(u_n, y) + \langle D x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of  $F_1$  that

$$\langle D x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n), \quad \forall y \in C$$

and

$$\langle D x_{n_k}, y - u_{n_k} \rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq F_1(y, u_{n_k}), \quad \forall y \in C. \quad (3.20)$$

Since  $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$ , and  $x_n \rightharpoonup w$ , it is easy to observe that  $u_{n_k} \rightarrow w$ . For any  $0 < t \leq 1$  and  $y \in C$ , let  $y_t = ty + (1-t)w$ , and we have  $y_t \in C$ . Then from (3.20), we obtain

$$\begin{aligned} \langle D y_t, y_t - u_{n_k} \rangle &\geq \langle D y_t, y_t - u_{n_k} \rangle - \langle D x_{n_k}, y_t - u_{n_k} \rangle - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + F_1(y_t, u_{n_k}) \\ &= \langle D y_t - D u_{n_k}, y_t - u_{n_k} \rangle + \langle D u_{n_k} - D x_{n_k}, y_t - u_{n_k} \rangle \\ &\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + F_1(y_t, u_{n_k}). \end{aligned} \quad (3.21)$$

Since  $D$  is Lipschitz continuous and  $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$ , we obtain  $\lim_{k \rightarrow \infty} \|D u_{n_k} - D x_{n_k}\| = 0$ . From the monotonicity of  $D$  and  $u_{n_k} \rightarrow w$ , it follows from (3.21) that

$$\langle D y_t, y_t - w \rangle \geq F_1(y_t, w). \quad (3.22)$$

Hence, from assumptions (i)-(iv) of Lemma 2.2 and (3.22), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) \leq t F_1(y_t, y) + (1-t) F_1(y_t, w) \\ &\leq t F_1(y_t, y) + (1-t) \langle D y_t, y_t - w \rangle \\ &\leq t F_1(y_t, y) + (1-t) t \langle D y_t, y - w \rangle, \end{aligned} \quad (3.23)$$

which implies that  $F_1(y_t, y) + (1-t) \langle D y_t, y - w \rangle \geq 0$ . Letting  $t \rightarrow 0_+$ , we have

$$F_1(w, y) + \langle D w, y - w \rangle \geq 0, \quad \forall y \in C,$$

which implies that  $w \in MEP(F_1)$ .



Furthermore, we show that  $w \in \Omega^*$ . Let

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where  $N_C v := \{w \in H : \langle w, v - u \rangle \geq 0, \forall u \in C\}$  is the normal cone to  $C$  at  $v \in C$ . Then  $T$  is maximal monotone and  $0 \in Tv$  if and only if  $v \in \Omega^*$  (see [28]). Let  $G(T)$  denote the graph of  $T$ , and let  $(v, u) \in G(T)$ ; since  $u - Av \in N_C v$  and  $z_n \in C$ , we have

$$\langle v - z_n, u - Av \rangle \geq 0. \quad (3.24)$$

On the other hand, it follows from  $z_n = P_C[u_n - \lambda_n A u_n]$  and  $v \in C$  that

$$\langle v - z_n, z_n - (u_n - \lambda_n A u_n) \rangle \geq 0$$

and

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + A u_n \right\rangle \geq 0.$$

Therefore, from (3.24) and the inverse strong monotonicity of  $A$ , we have

$$\begin{aligned} \langle v - z_{n_k}, u \rangle &\geq \langle v - z_{n_k}, Av \rangle \\ &\geq \langle v - z_{n_k}, Av \rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + A u_{n_k} \right\rangle \\ &\geq \langle v - z_{n_k}, Av - A z_{n_k} \rangle + \langle v - z_{n_k}, A z_{n_k} - A u_{n_k} \rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} \right\rangle \\ &\geq \langle v - z_{n_k}, A z_{n_k} - A u_{n_k} \rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} \right\rangle. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \|u_n - z_n\| = 0$  and  $u_{n_k} \rightarrow w$ , it is easy to observe that  $z_{n_k} \rightarrow w$ . Hence, we obtain  $\langle v - w, u \rangle \geq 0$ . Since  $T$  is maximal monotone, we have  $w \in T^{-1}0$ , and hence  $w \in \Omega^*$ . Thus we have

$$w \in \Omega^* \cap MEP(F_1) \cap F(T).$$

Observe that the constants satisfy  $0 \leq \rho\tau < \nu$  and

$$\begin{aligned} k \geq \eta &\iff k^2 \geq \eta^2 \\ &\iff 1 - 2\mu\eta + \mu^2 k^2 \geq 1 - 2\mu\eta + \mu^2 \eta^2 \\ &\iff \sqrt{1 - \mu(2\eta - \mu k^2)} \geq 1 - \mu\eta \\ &\iff \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \\ &\iff \mu\eta \geq \nu, \end{aligned}$$

therefore, from Lemma 2.5, the operator  $\mu F - \rho U$  is  $\mu\eta - \rho\tau$  strongly monotone, and we get the uniqueness of the solution of the variational inequality (3.19) and denote it by  $z \in \Omega^* \cap MEP(F_1) \cap F(T)$ .

Next, we claim that  $\limsup_{n \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_n - z \rangle \leq 0$ . Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_n - z \rangle &= \limsup_{k \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_{n_k} - z \rangle \\ &= \langle \rho U(z) - \mu F(z), w - z \rangle \leq 0. \end{aligned}$$

Next, we show that  $x_n \rightarrow z$ . We have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \langle P_C(V_n) - z, x_{n+1} - z \rangle \\ &= \langle P_C(V_n) - V_n, P_C(V_n) - z \rangle + \langle V_n - z, x_{n+1} - z \rangle \\ &\leq \langle \alpha_n(\rho U(x_n) - \mu F(z)) + (I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)(T(z)), x_{n+1} - z \rangle \\ &= \langle \alpha_n \rho(U(x_n) - U(z)), x_{n+1} - z \rangle + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + \langle (I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)(T(z)), x_{n+1} - z \rangle \\ &\leq \alpha_n \rho \tau \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n v) \|y_n - z\| \|x_{n+1} - z\| \\ &\leq \alpha_n \rho \tau \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n v) \{ \beta_n \|Sx_n - Sz\| + \beta_n \|Sz - z\| + (1 - \beta_n) \|z_n - z\| \} \|x_{n+1} - z\| \\ &\leq \alpha_n \rho \tau \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n v) \{ \beta_n \|x_n - z\| + \beta_n \|Sz - z\| + (1 - \beta_n) \|x_n - z\| \} \|x_{n+1} - z\| \\ &= (1 - \alpha_n(v - \rho\tau)) \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n v) \beta_n \|Sz - z\| \|x_{n+1} - z\| \\ &\leq \frac{1 - \alpha_n(v - \rho\tau)}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + (1 - \alpha_n v) \beta_n \|Sz - z\| \|x_{n+1} - z\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \frac{1 - \alpha_n(v - \rho\tau)}{1 + \alpha_n(v - \rho\tau)} \|x_n - z\|^2 + \frac{2\alpha_n}{1 + \alpha_n(v - \rho\tau)} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\ &\quad + \frac{2(1 - \alpha_n v) \beta_n}{1 + \alpha_n(v - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \\ &\leq (1 - \alpha_n(v - \rho\tau)) \|x_n - z\|^2 + \frac{2\alpha_n(v - \rho\tau)}{1 + \alpha_n(v - \rho\tau)} \\ &\quad \times \left\{ \frac{1}{v - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + \frac{(1 - \alpha_n v) \beta_n}{\alpha_n(v - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\}. \end{aligned}$$

Let  $\gamma_n = \alpha_n(v - \rho\tau)$  and  $\delta_n = \frac{2\alpha_n(v - \rho\tau)}{1 + \alpha_n(v - \rho\tau)} \left\{ \frac{1}{v - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + \frac{(1 - \alpha_n v) \beta_n}{\alpha_n(v - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\}$ .

We have

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and

$$\limsup_{n \rightarrow \infty} \left\{ \frac{1}{v - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + \frac{(1 - \alpha_n v) \beta_n}{\alpha_n (v - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\} \leq 0.$$

It follows that

$$\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0.$$

Thus all the conditions of Lemma 2.7 are satisfied. Hence we deduce that  $x_n \rightarrow z$ . This completes the proof.  $\square$

#### 4 Applications

In this section, we obtain the following results by using a special case of the proposed method for example.

Putting  $A = 0$  in Algorithm 3.1, we obtain the following result which can be viewed as an extension and improvement of the method of Wang and Xu [24] for finding the approximate element of the common set of solutions of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.

**Corollary 4.1** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $D : C \rightarrow H$  be  $\theta$ -inverse strongly monotone mappings. Let  $F_1 : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying assumptions (i)-(iv) of Lemma 2.2 and  $S, T : C \rightarrow C$  be a nonexpansive mappings such that  $F(T) \cap \text{MEP}(F_1) \neq \emptyset$ . Let  $F : C \rightarrow C$  be a  $k$ -Lipschitzian mapping and be  $\eta$ -strongly monotone, and let  $U : C \rightarrow C$  be a  $\tau$ -Lipschitzian mapping. For an arbitrary given  $x_0 \in C$ , let the iterative sequences  $\{u_n\}$ ,  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated by*

$$F_1(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C;$$

$$y_n = \beta_n Sx_n + (1 - \beta_n)u_n;$$

$$x_{n+1} = P_C [\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0,$$

where  $\{r_n\} \subset (0, 2\theta)$ ,  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\beta_n\} \subset (0, 1)$ . Suppose that the parameters satisfy  $0 < \mu < \frac{2\eta}{k^2}$ ,  $0 \leq \rho\tau < v$ , where  $v = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ . Also,  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{r_n\}$  are sequences satisfying conditions (a)-(d) of Algorithm 3.1. The sequence  $\{x_n\}$  converges strongly to  $z$ , which is the unique solution of the variational inequality

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in \text{MEP}(F_1) \cap F(T).$$

Putting  $U = f$ ,  $F = I$ ,  $\rho = \mu = 1$ , and  $A = 0$ , we obtain an extension and improvement of the method of Yao et al. [12] for finding the approximate element of the common set of solutions

of a mixed equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.

**Corollary 4.2** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $D : C \rightarrow H$  be  $\theta$ -inverse strongly monotone mappings. Let  $F_1 : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying assumptions (i)-(iv) of Lemma 2.2 and  $S, T : C \rightarrow C$  be a nonexpansive mappings such that  $F(T) \cap \text{MEP}(F_1) \neq \emptyset$ . Let  $f : C \rightarrow C$  be a  $\tau$ -Lipschitzian mapping. For an arbitrary given  $x_0 \in C$ , let the iterative sequences  $\{u_n\}$ ,  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated by*

$$F_1(u_n, y) + \langle Dx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C;$$

$$y_n = \beta_n Sx_n + (1 - \beta_n)u_n;$$

$$x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)T(y_n)], \quad \forall n \geq 0,$$

where  $\{r_n\} \subset (0, 2\theta)$ ,  $\{\alpha_n\}$ ,  $\{\beta_n\}$  are sequences in  $(0, 1)$  satisfying conditions (a)-(d) of Algorithm 3.1. The sequence  $\{x_n\}$  converges strongly to  $z$ , which is the unique solution of the variational inequality

$$\langle f(z) - z, x - z \rangle \leq 0, \quad \forall x \in \text{MEP}(F_1) \cap F(T).$$

**Remark 4.1** Some existing methods (e.g., [12, 14, 16, 17, 25]) can be viewed as special cases of Algorithm 3.1. Therefore, the new algorithm is expected to be widely applicable.

To verify the theoretical assertions, we consider the following example.

**Example 4.1** Let  $\alpha_n = \frac{1}{3n}$ ,  $\beta_n = \frac{1}{n^3}$ ,  $\lambda_n = \frac{1}{2(n+1)}$ , and  $r_n = \frac{n}{n+1}$ . We have

$$\lim_{n \rightarrow \infty} \alpha_n = \frac{1}{3} \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

and

$$\sum_{n=1}^{\infty} \alpha_n = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

The sequence  $\{\alpha_n\}$  satisfies condition (a).

$$\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = \lim_{n \rightarrow \infty} \frac{3}{n^2} = 0.$$

Condition (b) is satisfied. We compute

$$\alpha_{n-1} - \alpha_n = \frac{1}{3} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{3n(n-1)}.$$

It is easy to show  $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ . Similarly, we can show  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ . The sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy condition (c). We have

$$\liminf_{n \rightarrow \infty} r_n = \liminf_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} |r_{n-1} - r_n| &= \sum_{n=1}^{\infty} \left| \frac{n-1}{n} - \frac{n}{n+1} \right| \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &< \infty.\end{aligned}$$

Then, the sequence  $\{r_n\}$  satisfies condition (d). We compute

$$\begin{aligned}\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| &= \sum_{n=1}^{\infty} \left| \frac{1}{2n} - \frac{1}{2(n+1)} \right| \\ &= \frac{1}{2} \\ &< \infty.\end{aligned}$$

Then, the sequence  $\{\lambda_n\}$  satisfies condition (e).

Let  $\mathbb{R}$  be the set of real numbers,  $D = 0$ , and let the mapping  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$Ax = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$T(x) = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping  $F : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$F(x) = \frac{2x+5}{7}, \quad \forall x \in \mathbb{R},$$

let the mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$S(x) = \frac{x}{2}, \quad \forall x \in \mathbb{R},$$

let the mapping  $U : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$U(x) = \frac{x}{14}, \quad \forall x \in \mathbb{R},$$

and let the mapping  $F_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$F_1(x, y) = -3x^2 + xy + 2y^2, \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}.$$

It is easy to show that  $A$  is a 1-inverse strongly monotone mapping,  $T$  and  $S$  are non-expansive mappings,  $F$  is a 1-Lipschitzian mapping and  $\frac{1}{7}$ -strongly monotone and  $U$  is

**Table 1** The values of  $\{u_n\}$ ,  $\{z_n\}$ ,  $\{y_n\}$ , and  $\{x_n\}$  with initial values  $x_1 = 30$  and  $x_1 = -30$

	$x_1 = 30$				$x_1 = -30$			
	$u_n$	$z_n$	$y_n$	$x_n$	$u_n$	$z_n$	$y_n$	$x_n$
$n = 1$	8.571429	4.285714	15.000000	30.000000	-8.571429	-4.285714	-15.000000	-30.000000
$n = 2$	1.710361	0.427590	0.837364	7.411565	-1.726060	-0.431515	-0.845050	-7.479592
$n = 3$	0.085202	0.000000	0.007495	0.404709	-0.093178	0.000000	-0.008196	-0.442594
$n = 4$	-0.001479	0.000370	0.000306	-0.007393	-0.003130	0.000783	0.000648	-0.015652
$n = 5$	-0.001617	0.000808	0.000769	-0.008354	-0.001585	0.000792	0.000753	-0.008187
$n = 6$	-0.001215	0.000911	0.000892	-0.006422	-0.001216	0.000912	0.000893	-0.006430
$n = 7$	-0.000972	0.000972	0.000962	-0.005226	-0.000972	0.000972	0.000962	-0.005225
$n = 8$	-0.000805	0.001006	0.000999	-0.004380	-0.000805	0.001006	0.000999	-0.004380
$n = 9$	-0.000682	0.001024	0.001020	-0.003754	-0.000682	0.001024	0.001020	-0.003754
$n = 10$	-0.000590	0.001032	0.001030	-0.003271	-0.000590	0.001032	0.001030	-0.003271

$\frac{1}{7}$ -Lipschitzian. It is clear that

$$\Omega^* \cap MEP(F_1) \cap F(T) = \{0\}.$$

By the definition of  $F_1$ , we have

$$\begin{aligned} 0 &\leq F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ &= -3u_n^2 + u_n y + 2y^2 + \frac{1}{r_n} (y - u_n)(u_n - x_n). \end{aligned}$$

Then

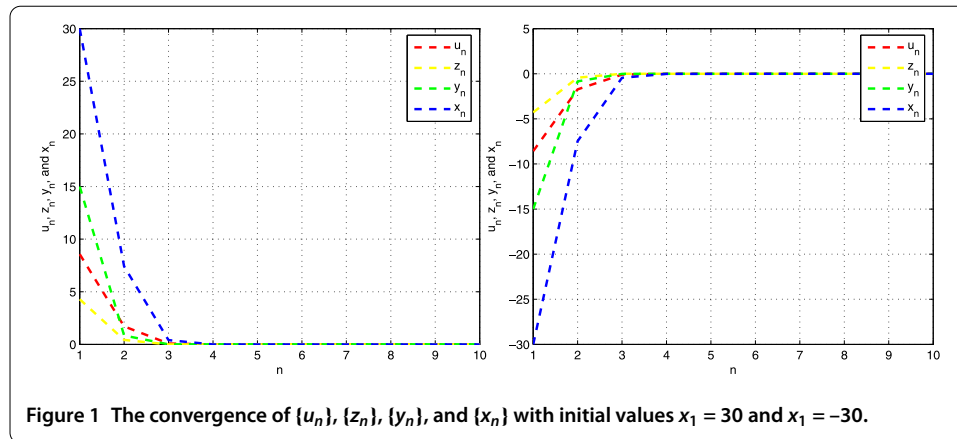
$$\begin{aligned} 0 &\leq r_n(-3u_n^2 + u_n y + 2y^2) + (y u_n - y x_n - u_n^2 + u_n x_n) \\ &= 2r_n y^2 + (r_n u_n + u_n - x_n) y - 3r_n u_n^2 - u_n^2 + u_n x_n. \end{aligned}$$

Let  $B(y) = 2r_n y^2 + (r_n u_n + u_n - x_n) y - 3r_n u_n^2 - u_n^2 + u_n x_n$ .  $B(y)$  is a quadratic function of  $y$  with coefficient  $a = 2r_n$ ,  $b = r_n u_n + u_n - x_n$ ,  $c = -3r_n u_n^2 - u_n^2 + u_n x_n$ . We determine the discriminant  $\Delta$  of  $B$  as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (r_n u_n + u_n - x_n)^2 - 8r_n(-3r_n u_n^2 - u_n^2 + u_n x_n) \\ &= u_n^2 + 10r_n u_n^2 + 25u_n^2 r_n^2 - 2x_n u_n - 10x_n u_n r_n + x_n^2 \\ &= (u_n + 5u_n r_n)^2 - 2x_n(u_n + 5u_n r_n) + x_n^2 \\ &= (u_n + 5u_n r_n - x_n)^2. \end{aligned}$$

We have  $B(y) \geq 0$ ,  $\forall y \in \mathbb{R}$ . If it has at most one solution in  $\mathbb{R}$ , then  $\Delta = 0$ , we obtain

$$u_n = \frac{x_n}{1 + 5r_n}. \quad (4.1)$$



**Figure 1** The convergence of  $\{u_n\}$ ,  $\{z_n\}$ ,  $\{y_n\}$ , and  $\{x_n\}$  with initial values  $x_1 = 30$  and  $x_1 = -30$ .

For every  $n \geq 1$ , from (4.1), we rewrite (3.1) as follows:

$$\begin{cases} z_n = \frac{x_n}{1+5r_n} - \frac{x_n}{4(n+1)(1+5r_n)}; \\ y_n = \frac{x_n}{2n^3} + (1 - \frac{1}{n^3})z_n; \\ x_{n+1} = \rho \frac{x_n}{42n} + \frac{y_n}{2} - \mu \frac{y_n+5}{21n}. \end{cases}$$

In all the tests we take  $\rho = \frac{1}{15}$  and  $\mu = \frac{1}{7}$ . In our example,  $\eta = \frac{1}{7}$ ,  $k = 1$ ,  $\tau = \frac{1}{7}$ . It is easy to show that the parameters satisfy  $0 < \mu < \frac{2\eta}{k^2}$ ,  $0 \leq \rho\tau < v$ , where  $v = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ . All codes were written in Matlab, the values of  $\{u_n\}$ ,  $\{z_n\}$ ,  $\{y_n\}$ , and  $\{x_n\}$  with different  $n$  are reported in Table 1.

**Remark 4.2** Table 1 and Figure 1 show that the sequences  $\{u_n\}$ ,  $\{z_n\}$ ,  $\{y_n\}$ , and  $\{x_n\}$  converge to 0, where  $\{0\} = \Omega^* \cap MEP(F_1) \cap F(T)$ .

#### Competing interests

The author declares that he has no competing interests.

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#### References

- Lions, JL, Stampacchia, G: Variational inequalities. *Commun. Pure Appl. Math.* **20**, 493-512 (1967)
- Zhou, H: Convergence theorems of fixed points for  $k$ -strict pseudo-contractions in Hilbert spaces. *Nonlinear Anal.* **69**, 456-462 (2008)
- Moudafi, A, Théra, MM: Proximal and Dynamical Approaches to Equilibrium Problems. *Lecture Notes in Economics and Mathematical Systems*, vol. 477. Springer, New York (1999)
- Moudafi, A: Mixed equilibrium problems sensitivity analysis and algorithmic aspect. *Comput. Math. Appl.* **44**, 1099-1108 (2002)
- Chang, SS, Joseph Lee, HW, Chan, CK: A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. *Nonlinear Anal.* **70**, 3307-3319 (2009)
- Katchang, P, Kumam, P: A new iterative algorithm for equilibrium problems, variational inequalities and fixed point problems in a Hilbert space. *Appl. Math. Comput.* **32**, 19-38 (2010)
- Plubtieng, S, Punpaeng, R: A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **336**, 455-469 (2007)
- Qin, X, Shang, M, Su, Y: A general iterative method for equilibrium problem and fixed point problem in Hilbert spaces. *Nonlinear Anal.* **69**, 3897-3909 (2008)
- Suwannaut, S, Kangtunyakarn, A: The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem. *Fixed Point Theory Appl.* **2013**, 291 (2013)
- Combettes, PL, Hirstoaga, SA: Equilibrium programming using proximal like algorithms. *Math. Program.* **78**, 29-41 (1997)
- Gu, G, Wang, S, Cho, YJ: Strong convergence algorithms for hierarchical fixed points problems and variational inequalities. *J. Appl. Math.* **2011**, 1-17 (2011)

12. Yao, Y, Cho, YJ, Liou, YC: Iterative algorithms for hierarchical fixed points problems and variational inequalities. *Math. Comput. Model.* **52**(9-10), 1697-1705 (2010)
13. Moudafi, A: Krasnoselski-Mann iteration for hierarchical fixed-point problems. *Inverse Probl.* **23**(4), 1635-1640 (2007)
14. Mainge, PE, Moudafi, A: Strong convergence of an iterative method for hierarchical fixed-point problems. *Pac. J. Optim.* **3**(3), 529-538 (2007)
15. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **318**(1), 43-52 (2006)
16. Cianciaruso, F, Marino, G, Muglia, L, Yao, Y: On a two-steps algorithm for hierarchical fixed point problems and variational inequalities. *J. Inequal. Appl.* **2009**, 1-13 (2009)
17. Ceng, LC, Anasri, QH, Yao, JC: Some iterative methods for finding fixed points and for solving constrained convex minimization problems. *Nonlinear Anal.* **74**, 5286-5302 (2011)
18. Bnouhachem, A: A modified projection method for a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed point problem. *Fixed Point Theory Appl.* **2014**(22), 1-25 (2014)
19. Bnouhachem, A: Strong convergence algorithm for split equilibrium problems and hierarchical fixed point problems. *Sci. World J.* **2014**, Article ID 390956 (2014)
20. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **63**, 123-145 (1994)
21. Yao, Y, Liou, YC, Kang, SM: Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. *Comput. Math. Appl.* **59**(11), 3472-3480 (2010)
22. Suzuki, N: Moudafi's viscosity approximations with Meir-Keeler contractions. *J. Math. Anal. Appl.* **325**, 342-352 (2007)
23. Xu, HK: Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66**, 240-256 (2002)
24. Wang, Y, Xu, W: Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities. *Fixed Point Theory Appl.* **121**(1), 1-9 (2013)
25. Tian, M: A general iterative algorithm for nonexpansive mappings in Hilbert spaces. *Nonlinear Anal.* **73**, 689-694 (2010)
26. Marino, G, Xu, HK: Explicit hierarchical fixed point approach to variational inequalities. *J. Optim. Theory Appl.* **149**(1), 61-78 (2011)
27. Acedo, GL, Xu, HK: Iterative methods for strictly pseudo-contractions in Hilbert space. *Nonlinear Anal.* **67**, 2258-2271 (2007)
28. Rockafellar, RT: On the maximality of sums nonlinear monotone operators. *Trans. Am. Math. Soc.* **149**, 75-88 (1970)

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